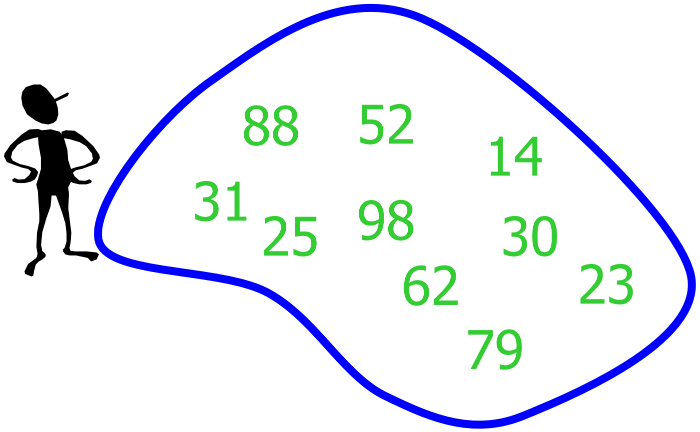
Algorithms Analysis

Lecture 6
Quicksort

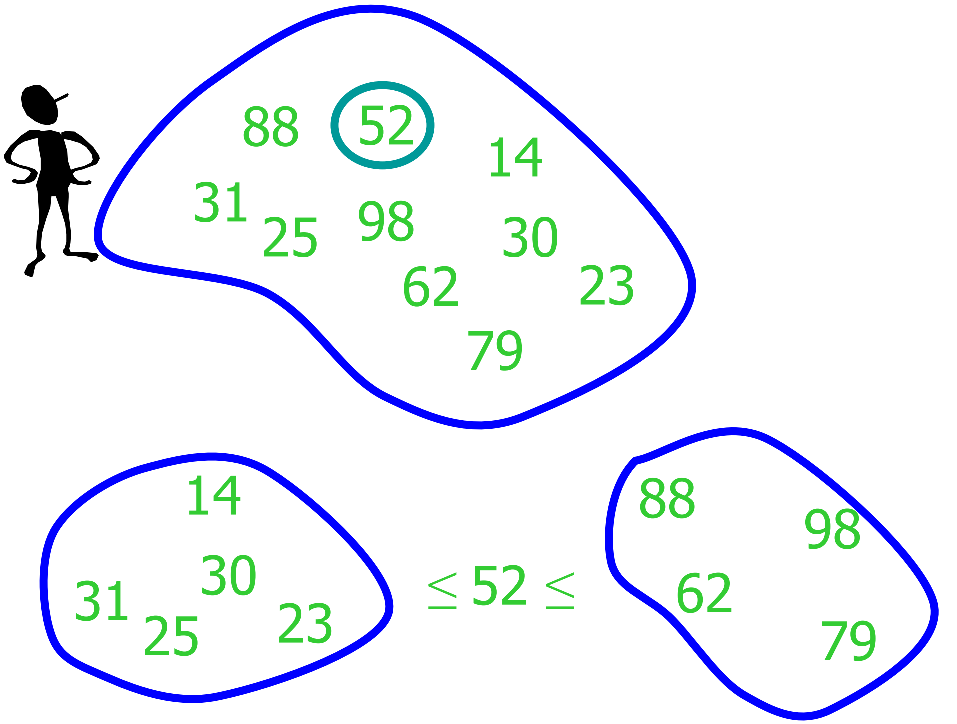
Quick Sort



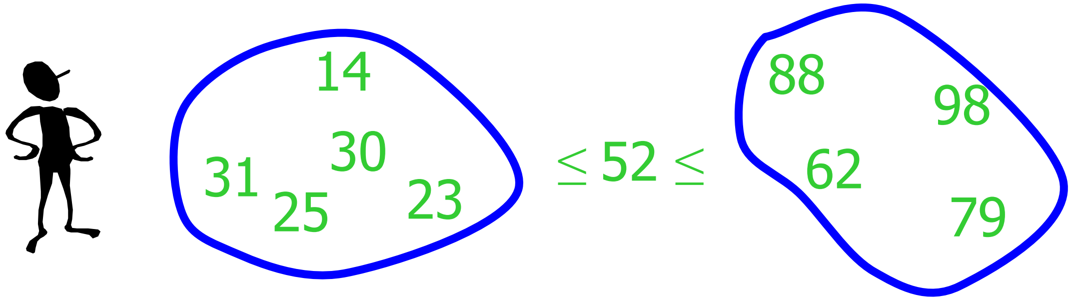
Divide and Conquer

Quick Sort

Partition set into two using randomly chosen pivot



Quick Sort



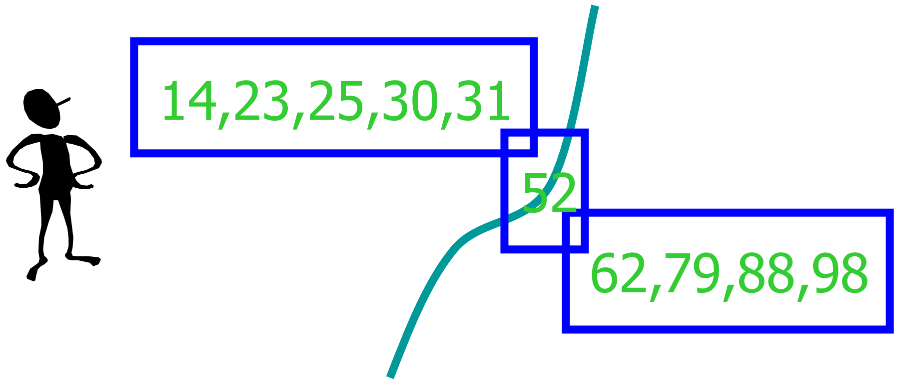
sort the first half.

$\boxed{14,23,25,30,31}$

sort the second half.

$\boxed{62,79,98,88}$

Quick Sort



Glue pieces together.

14,23,25,30,31,52,62,79,88,98

Quicksort

- Quicksort pros [advantage]:
  - Sorts in place
  - Sorts $O(n \lg n)$ in the average case
  - Very efficient in practice , it’s quick

- Quicksort cons [disadvantage]:
  - Sorts $O(n^2)$ in the worst case
  - And the worst case doesn’t happen often … sorted

Quicksort

- Another divide-and-conquer algorithm:
- *Divide*: $A[p \ldots r]$ is partitioned (rearranged) into two nonempty subarrays $A[p \ldots q\text{-}1]$ and $A[q\text{+}1 \ldots r]$ s.t. each element of $A[p \ldots q\text{-}1]$ is less than or equal to each element of $A[q\text{+}1 \ldots r]$. Index q is computed here, called **pivot**.
- *Conquer*: two subarrays are sorted by recursive calls to quicksort.
- *Combine*: unlike merge sort, no work needed since the subarrays are sorted in place already.

Quicksort

- The basic algorithm to sort an array *A* consists of the following four easy steps:

  – If the number of elements in *A* is 0 or 1, then return
  – Pick any element ***v*** in *A*. This is called the ***pivot***

  – Partition *A*-{***v***} (the remaining elements in *A*) into two disjoint groups:
    - $A_1 = \{\boldsymbol{x} \in A\text{-}\{\boldsymbol{v}\} \mid \boldsymbol{x} \leq \boldsymbol{v}\}$, and
    - $A_2 = \{\boldsymbol{x} \in A\text{-}\{\boldsymbol{v}\} \mid \boldsymbol{x} \geq \boldsymbol{v}\}$
  – return
    - { quicksort(A_1) followed by ***v*** followed by quicksort(A_2)}

Quicksort

- Small instance has *n* ≤ 1
  - Every small instance is a sorted instance
- To sort a large instance:
  - select a pivot element from out of the ***n*** elements
- Partition the ***n*** elements into 3 groups left, middle and right
  - The middle group contains only the pivot element
  - All elements in the left group are ≤ pivot
  - All elements in the right group are ≥ pivot
- Sort left and right groups recursively
- Answer is sorted left group, followed by middle group followed by sorted right group

Quicksort Code

```
P: first element
r: last element
Quicksort(A, p, r)
{
    if (p < r)
    {
        q = Partition(A, p, r)
        Quicksort(A, p  , q-1)
        Quicksort(A, q+1      , r)
    }
}
```

- Initial call is **Quicksort**(*A*, 1, *n*), where *n* in the length of *A*

Partition

- Clearly, all the action takes place in the **`partition()`** function
  - Rearranges the subarray in place
  - End result:
    - Two subarrays
    - All values in first subarray $\leq$ all values in second
  - Returns the **index** of the “pivot” element separating the two subarrays

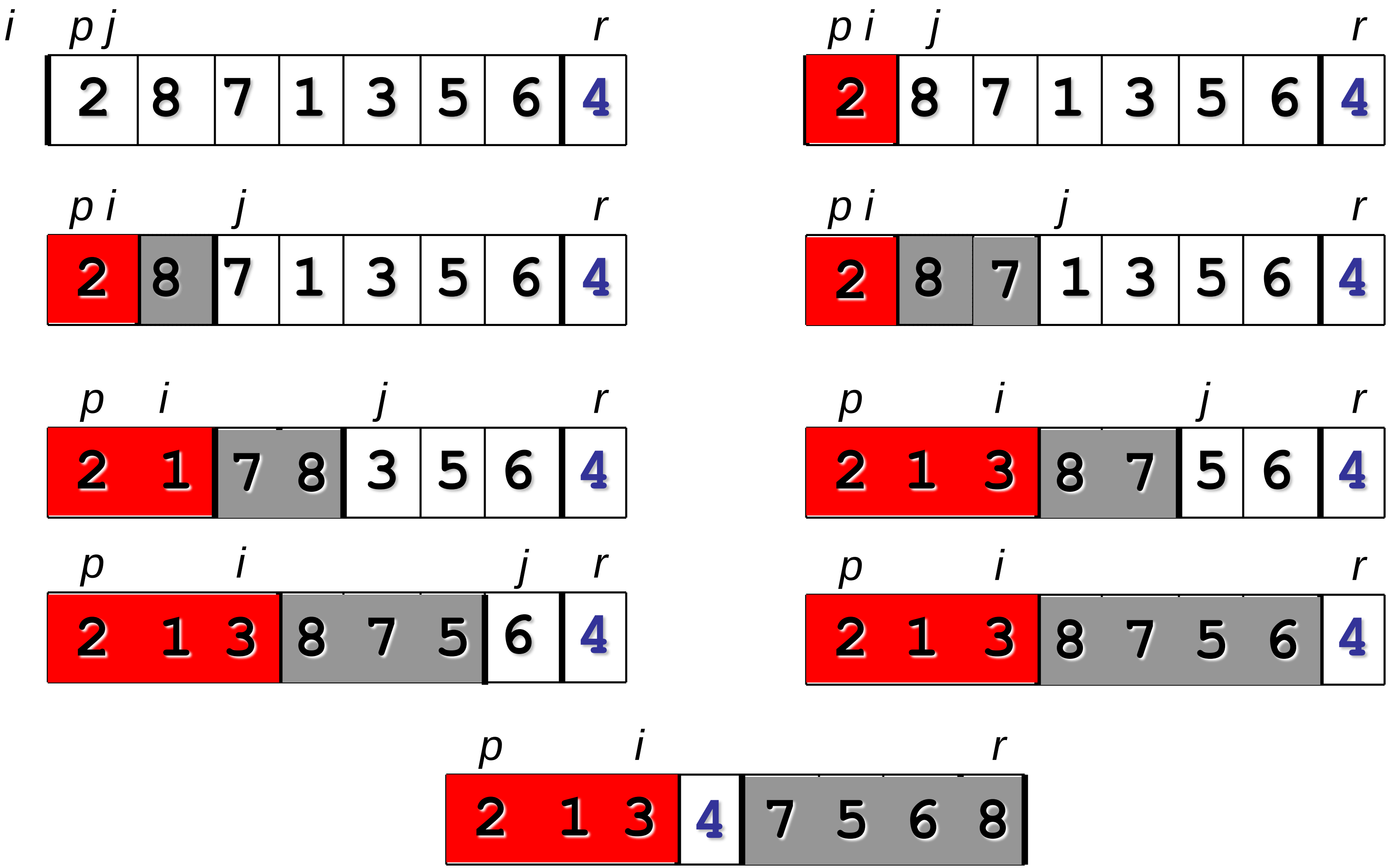
Partition Code

```
Partition(A, p, r)
{
    x = A[r]                // x is pivot
    i = p - 1
    for j = p to r – 1
    {
        do if A[j] <= x
            then
            {
              i = i + 1
              exchange A[i] ↔ A[j]
            }
    }
    exchange A[i+1] ↔ A[r]
    return i+1
}
```

partition() runs in O(n) time

Partition Example

$$A = \{2, 8, 7, 1, 3, 5, 6, 4\}$$



Partition Example Explanation

- Red shaded elements are in the first partition with values $\leq x$ (pivot)
- Gray shaded elements are in the second partition with values $\geq x$ (pivot)
- The unshaded elements have no yet been put in one of the first two partitions
- The final white element is the pivot

Choice Of Pivot

Three ways to choose the pivot:

- Pivot is **rightmost** element in list that is to be sorted
  - When sorting $A[6:20]$, use $A[20]$ as the pivot
  - Textbook implementation does this

- **Randomly** select one of the elements to be sorted as the pivot
  - When sorting $A[6:20]$, generate a random number r in the range [6, 20]
  - Use $A[r]$ as the pivot

Worst Case Partitioning

- The running time of quicksort depends on whether the partitioning is **balanced** or not.
- $\Theta(n)$ time to partition an array of n elements
- Let $T(n)$ be the time needed to sort n elements
- $T(0) = T(1) = c$, where c is a constant
- When $n > 1$,
  - $T(n) = T(|\text{left}|) + T(|\text{right}|) + \Theta(n)$
- $T(n)$ is maximum (worst-case) when <u>either |left| = 0 or |right| = 0 following each partitioning</u>

Worst Case Partitioning

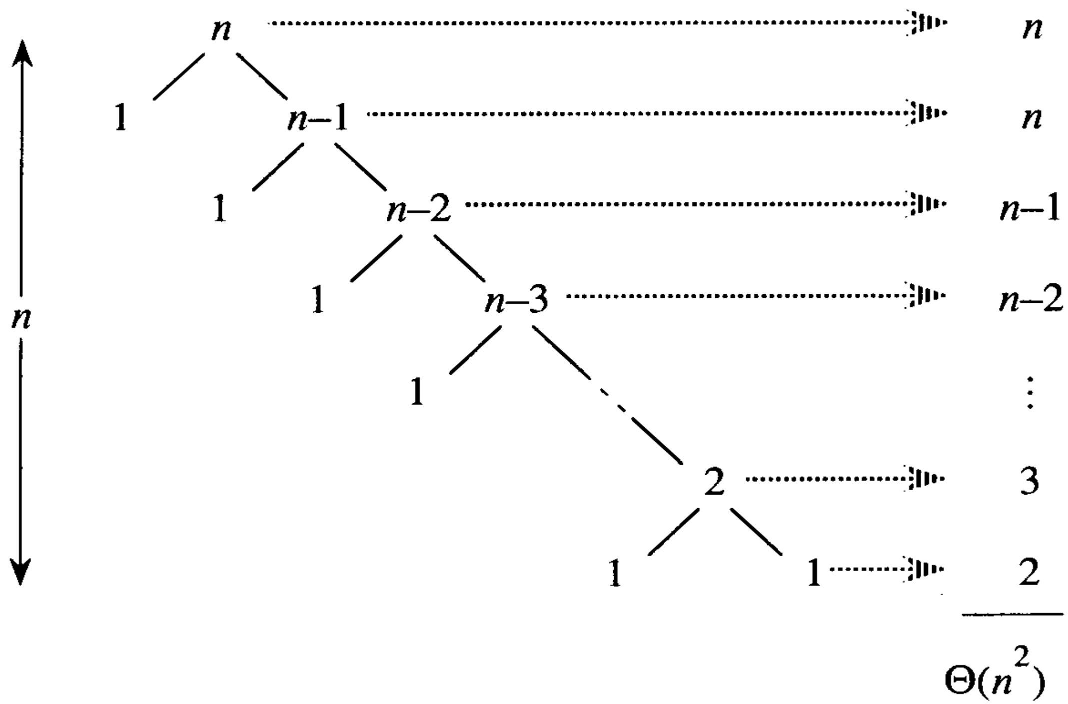


Figure 8.2 A recursion tree for QUICKSORT in which the PARTITION procedure always puts only a single element on one side of the partition (the worst case). The resulting running time is $\Theta(n^2)$.

Worst Case Partitioning

- **Worst-Case** Performance (unbalanced):
  - $T(n) = T(1) + T(n\text{-}1) + \Theta(n)$
    - partitioning takes $\Theta(n)$

  $= [2 + 3 + 4 + \ldots + n\text{-}1 + n\,] + n =$

  $= [\sum_{k\,=\,2\text{ to }n} k\,] + n = \Theta(n^2)$

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = n(n+1)/2 = \Theta(n^2)$$

- This occurs when
  - the input is **completely sorted**
- or when
  - the pivot is always the **smallest (largest)** element

Best Case Partition

- When the partitioning procedure produces two regions of size ***n*/2**, we get the a **balanced** partition with **best case** performance:
  - $T(n) = 2T(n/2) + \Theta(n) = \Theta(n \lg n)$

- **Average** complexity is also $\Theta(n \lg n)$

Best Case Partitioning

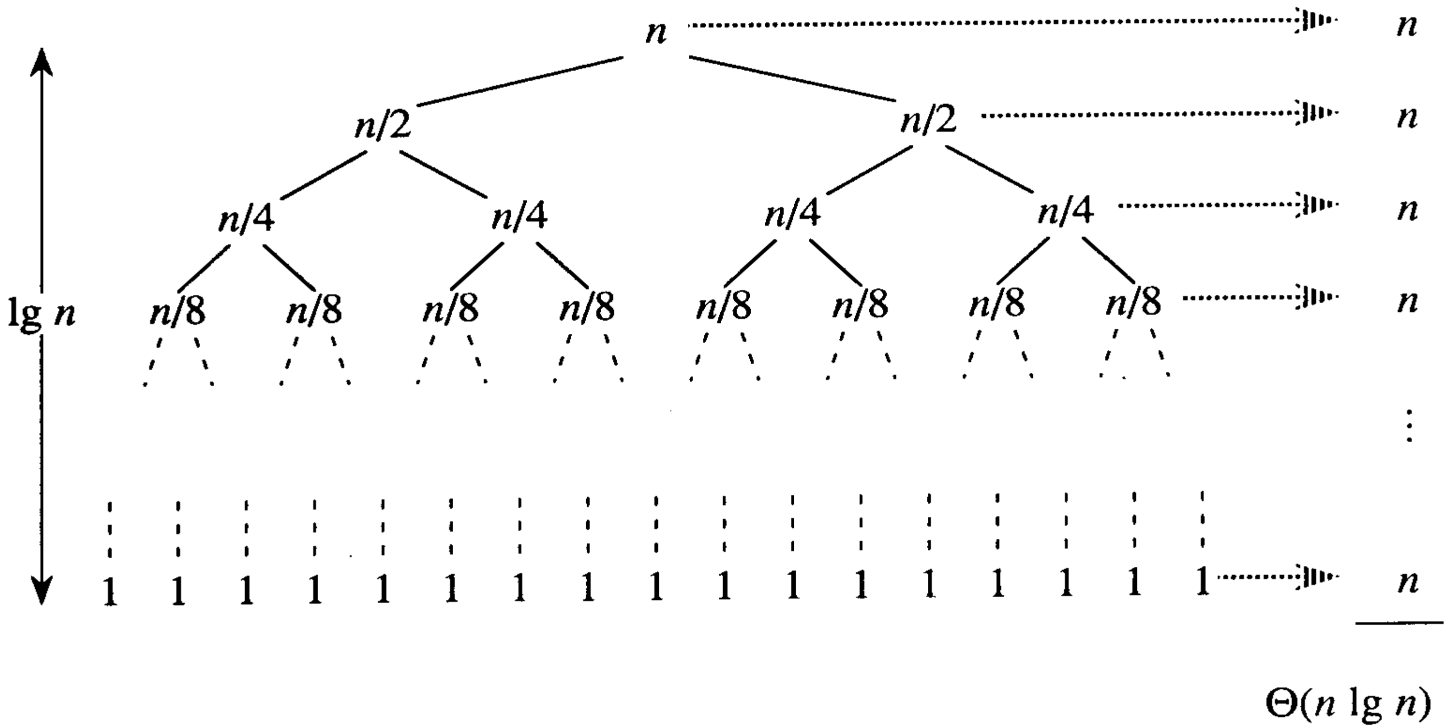


Figure 8.3 A recursion tree for QUICKSORT in which PARTITION always balances the two sides of the partition equally (the best case). The resulting running time is $\Theta(n \lg n)$.

Average Case

- Assuming random input, average-case running time is much closer to $\Theta(n \lg n)$ than $\Theta(n^2)$

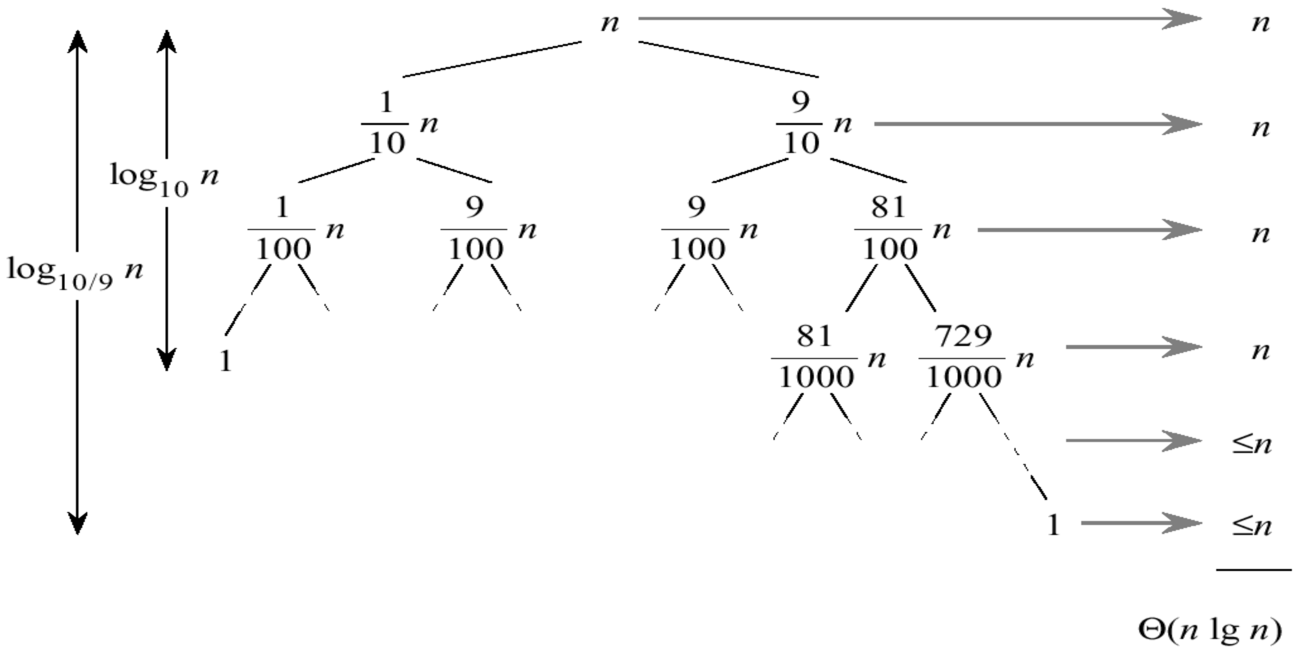
- First, a more intuitive explanation/example:
  - Suppose that **partition()** always produces a 9-to-1 **proportional** split. This looks quite unbalanced!
  - The recurrence is thus:

    $T(n) = T(9n/10) + T(n/10) + \Theta(n) = \Theta(n \lg n)$?

[Using recursion tree method to solve]

Average Case

$$T(n) = T(n/10) + T(9n/10) + \Theta(n) = \Theta(n \log n)!$$



$\log_2 n = \log_{10} n / \log_{10} 2$

Average Case

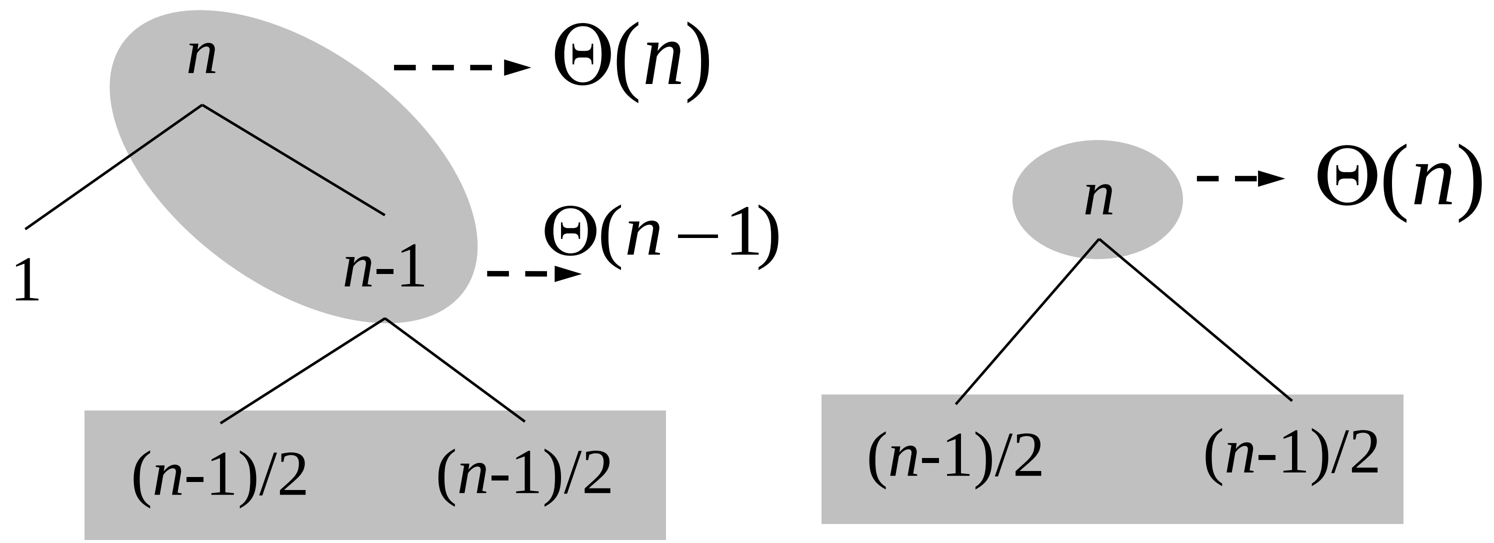
- Every level of the tree has cost cn, until a boundary condition is reached at depth $\log_{10} n = \Theta(\lg n)$, and then the levels have cost at most cn.

- The recursion terminates at depth $\log_{10/9} n = \Theta(\lg n)$.

- The total cost of quicksort is therefore $O(n \lg n)$.

Average Case

- What happens if we bad-split root node, then good-split the resulting size (n-1) node?
  - We end up with three subarrays, size
    - 1, (n-1)/2, (n-1)/2
  - Combined cost of splits = n + n-1 = 2n -1 = $\Theta(n)$



Intuition for the Average Case

- Suppose, we alternate lucky and unlucky cases to get an average behavior

$$L(n) = 2U(n/2) + \Theta(n) \quad \text{lucky}$$

$$U(n) = L(n-1) + \Theta(n) \quad \text{unlucky}$$

we consequently get

$$\begin{aligned} L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2-1) + \Theta(n) \\ &= \Theta(n \log n) \end{aligned}$$

The combination of good and bad splits would result in ***T*(*n*) = O (*n* lg *n*)**, but with slightly **larger constant** hidden by the O-notation.

Randomized Quicksort

- An algorithm is *randomized* if its behavior is determined not only by the input but also by values produced by a *random-number generator*.
- Exchange $A[r]$ with an element chosen at random from $A[p \ldots r]$ in **Partition**.
- This ensures that the pivot element is equally likely to be any of input elements.
- We can sometimes add randomization to an algorithm in order to obtain good average-case performance over all inputs.

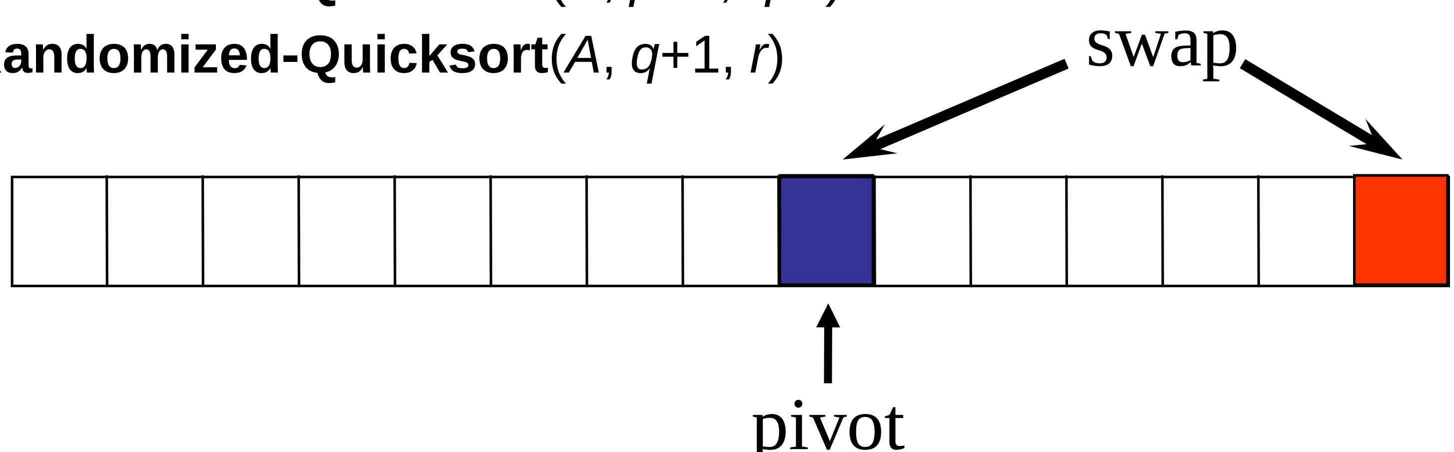
Randomized Quicksort

<u>Randomized-Partition(*A, p, r*)</u>

1. $i \leftarrow Random(p, r)$
2. exchange $A[r] \leftrightarrow A[i]$
3. **return Partition**(A, p, r)

<u>Randomized-Quicksort(*A, p, r*)</u>

1. **if** $p < r$
2. **then** $q \leftarrow$ **Randomized-Partition**(A, p, r)
3. **Randomized-Quicksort**$(A, p, q\text{-}1)$
4. **Randomized-Quicksort**$(A, q+1, r)$



Review: Analyzing Quicksort

- *What will be the worst case for the algorithm?*
  - Partition is always unbalanced
- *What will be the best case for the algorithm?*
  - Partition is balanced

Summary: Quicksort

- In worst-case, efficiency is $\Theta(n^2)$
  - But easy to avoid the worst-case
- On average, efficiency is $\Theta(n \lg n)$
- Better space-complexity than mergesort.
- In practice, runs fast and widely used